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Real-time Obstacles Avoidance for Vehicles in the Urban Grand Challenge

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In this paper, the real-time trajectory planning problem is considered for vehicles in the urban grand challenge. Typically, reference paths generated by a high level path planner are not feasible for vehicles and need improvement. In addition, moving obstacles in the environment are generally not known apriori, which also requires the paths to be able to be replanned in real-time. The proposed method presents three novel features to satisfy those requirements. First, all the paths satisfying boundary conditions and the vehicle's kinematic constraints are parameterized in terms of polynomials of sufficient order. Then, a collision-free criterion is developed for avoiding "hard" and "soft" obstacles detected. In the third step, a L_2-norm performance index is introduced to find the best path among the class of collision-free paths. The performance index is chosen such that a path analogous to the shortest path can be analytically solved. The proposed method provides a systematic and analytical solution to find the feasible path while addressing obstacle avoidance and guaranteeing performance. Simulation results illustrate the proposed algorithm.

Nomenclature

(x, y)	Cartesian coordinates of the guidepoint
$[a_0\ a_1\ \cdots\ a_p]$	Coefficients of the pth polynomial
$\boldsymbol{q}$	Configuration space variables

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Ω	Collision-free intervals
ϕ	Steering angle
ρ	Driving wheel radius
ρ_o	Minimum measure on physical envelope of the vehicle
$\mathbf{u}$	Control inputs
θ	Orientation of the car body
g_2, $g_{1,i}$, $g_{0,i}$	Coefficients of geometry constraints in terms of second order polynomials
G_i	Geometry constraints
H	Motion constraints
i	Index of the obstacle
J	Performance index
k	Index of the sampling period
l	Distance between the two wheel axles
N_o	Numbers of obstacles in the environment
n_o	Numbers of obstacles detected
N_q	Dimensions of the configuration space
O_i	Center location of obstacles
p	Order of the polynomial
r_0	Radius of the vehicle's physical envelope
r_i	Radius of obstacles
R_s	Vehicle's Sensing range
t_0	Start time
t_f	End time
T_s	Sampling period
u_1	Angular velocity of the driving wheel
u_2	Steering rate of the driving wheel
v	Operational velocity
v_i	Instantaneous velocity of obstacles
w_1	Vehicle's horizontal speed

I. Introduction

TEAMUCF and the Knight Rider vehicle were conceived over three years ago at the beginning of the 2005 DARPA Grand Challenge event. With the inception of the DARPA Urban Challenge,TeamUCF has built on the existing capabilities of the Knight Rider vehicle, creating a competition vehicle with the goal of meeting all the mission objectives of DARPA and performing at a level that challenges all other entries. Our Urban Challenge vehicle is known as Knight Rider VIP and focuses on the implementation of higher-level capabilities through the use of Vision, Intelligence, and Planning in a technical and motivational sense. Computer vision systems utilize a combination of sensors including active laser scanners, visible spectrum passive cameras, and near field acoustic sensors. Intelligence systems include a context-based reasoning subsystem which builds a real-time environmental model augmenting the supplied Route Network Definition File (RNDF) and develops mission plans to achieve objectives identified in the Mission Definition File (MDF). Planning provides tactical guidance information and vehicle control to achieve the defined missions in a safe and effective manner.

The most interesting element of planning is perhaps the generation of dense path information from the relatively sparse information provided by Intelligence. The fundamental objective of this element is to develop a real-time path planning and control algorithm by which the vehicle can autonomously navigate through a dynamically changing and congested environment. In the urban scenarios the vehicle must operate within very tight space, must take motion constraints into consideration (including boundary condition, kinematic constraints and velocity bounds), must avoid static as well as moving obstacles, and must perform complicated maneuvers whenever necessary.

A lot of research efforts have been devoted toward path planning in the presence of static obstacles, and many techniques have been proposed. Among them, the method of potential field is widely used,[1] and the basic idea is that, in planning a trajectory, potential fields are built around obstacles and pathways to expel the trajectory from obstacles and to bring the trajectory close to the final destination. Follow-up work can be found in some papers.[2–5] However, this methodology does not explicitly consider the system's kinematic constraints, and the generated trajectories may not be feasible. In order to solve this problem, some remedies have been proposed in the literature by first generating the paths without considering kinematic constraints and then accommodating the feasibility segmentally. For instance, Sundar and Shiller[6] gave the online suboptimal obstacle avoidance algorithm and Laumond *et al.*[7] and Reeds and Shepp[8] presented the nonholonomic path planner using a sequence of optimal path segments. Also popular are the spline methods,[9–11] in which a sequence of splines are used to generate a path through a given set of waypoints in the environment. However, the obtained trajectory is fixed and may not be collision-free for a dynamically changing environment. This makes it less interesting for real-time trajectory planning. Nonetheless, its underlining idea of parameterization and optimization is quite general and useful.

Exhaustive search and numerical iteration methods have also been used to deal with kinematic constraints and collision avoidance. By dividing the space into a series of regions, Barraquand and Latombe[12] found a safe path for the vehicle by starting at the initial condition and successively searching adjacent regions to the goal. The path segments are constructed by discretizing the controls and integrating the equations of motion. In Divelbiss and Wen's paper,[13] the obstacle avoidance criterion and kinematic model are typically converted into a set of inequality and equality constrains, and numerical iterations are used to approximate or determine the path that satisfies all the constraints.

Few results are available to deal with the moving obstacles. Kant and Zucker decomposed the dynamic motion planning to a static trajectory planning problem and a velocity planning problem.[14] Erdmann and Lozano-Perez treated the time as a state variable and recasted the dynamic motion planning problem into a static one,[15] and so on. However, these approaches suffer from incomplete information and their solutions are not guaranteed.

Recently, to solve the real-time trajectory planning problem for car-like vehicle moving in a dynamic environment, a new method is presented by Qu *et al.*[16] A class of collision-free trajectories in a closed form is analytically determined by one adjustable parameter. However, no optimal performance is considered in the solution. Thus the obtained trajectory may not be efficient in certain contexts. More recently, as a continuation of the work,[16] an optimal solution is obtained in Yang *et al.*[17] However it does not explicitly address the more realistic situation with different types of obstacles such as "hard" and "soft" obstacles, which may call for different criterion for more efficiently generating optimal trajectories.

This paper is organized as follows. In section II, the problem is rigorously formulated. Particularly, we first give the mathematical models for vehicle and dynamical environment. Then, the family of trajectories in terms of parameterized polynomials is presented. Third, an optimal performance index in the form of L_2 norm is defined for trajectory planning. Finally, we introduce the collision avoidance criterion. In section III, an analytical and optimal collision-free solution for both "hard" and "soft" obstacles are obtained, and the design process is illustrated step by step. Extensive simulations are performed to validate the effectiveness of the proposed approach in section IV. The conclusion is drawn in section V.

II. Problem Statement and Mathematical Formulation

In this paper, an optimal and real-time trajectory planning problem is formulated and solved. At the abstract level, the general problem of optimal trajectory planning is to determine $\boldsymbol{q}(t) : [t_0, t_f] \mapsto \mathbb{R}^{N_q}$ which meets the following sets of conditions:

C.1:Boundary conditions at the initial and terminal time instants, that is,

$$\boldsymbol{q}(t_0) = \boldsymbol{q}_0, \quad \boldsymbol{q}(t_f) = \boldsymbol{q}_f, \quad t_f > t_0 \tag{1}$$

C.2:Motion constraints in the form of

$$H(\dot{\boldsymbol{q}}(t), \boldsymbol{q}(t), \boldsymbol{u}(t)) = 0, \quad \forall t \in (t_0, t_f) \tag{2}$$

for some control inputs $\boldsymbol{u} \in \mathbb{R}^{N_u}$;

C.3:Minimization of performance index

$$J[\boldsymbol{q}(\cdot), \boldsymbol{u}(\cdot), t_0, t_f] = \int_{t_0}^{t_0+T} W(\boldsymbol{q}(t), \boldsymbol{u}(t))\,\mathrm{d}t \tag{3}$$

C.4:Collision avoidance, i.e., for any point $\boldsymbol{q}_{obs}(t)$ related to any of the obstacles,

$$\|\boldsymbol{q}(t) - \boldsymbol{q}_{obs}(t)\| > \rho_o \tag{4}$$

where N_q is the dimension of the configuration space, $T \triangleq t_f - t_0$, $\rho_0 > 0$ is the minimum measure on physical envelope of the vehicle, and $\| \cdot \|$ denotes an appropriate distance measure. ***Feasible*** trajectories are defined to be ones satisfying conditions **C.1**, **C.2** and **C.4**.

In real-world applications, the solution to a trajectory planning problem depends on specifics of the above conditions. Several of the conditions, such as boundary condition (1) and motion constraints (2), are usually given, nonetheless it is important that a good algorithm is developed to accommodate their variations in different applications. Optimal performance index (3) and collision avoidance criteria (4) are chosen by the designer to obtain the best algorithm possible.

In the rest of the section, specific choices of the conditions are made to illustrate the features and generality of the proposed trajectory planning algorithm. In particular, a kinematic model of a differentially driven vehicle is stated in subsection A and is used to illustrate how condition **C.2** is handled in the proposed algorithm. To describe most changing environments in the real world, a piecewise constant environment model is introduced in subsection B, and obstacle envelopes are specified. To handle a dynamically changing environment, the class of piecewise-constant and polynomial-parameterized trajectories are defined in subsection C. Our approach is to determine a real-time, optimal, and feasible trajectory in the class. To this end, condition **C.3** is chosen in subsection D to be a L_2-norm based performance index in order to obtain an analytical solution; and in subsection E, collision-avoidance criteria are presented to handle obstacles.

A. Kinematic Model of the Knight Rider Vehicle

The Knight Rider Vehicle is shown in Fig. 1, it is a front-steering and back-driving vehicle. Let (x, y) be the Cartesian coordinates of the guidepoint (i.e., the middle point of rear axle), θ be the orientation of the car body with respect to the x axis, and ϕ be the steering angle, ρ be the driving wheel radius, l be the distance between the two wheel axles, u_1 be the angular velocity of the driving wheel, and u_2 be the steering rate. By limiting $\phi \in (-\pi/2,\ \pi/2)$

Fig. 1 The Knight Rider Vehicle.

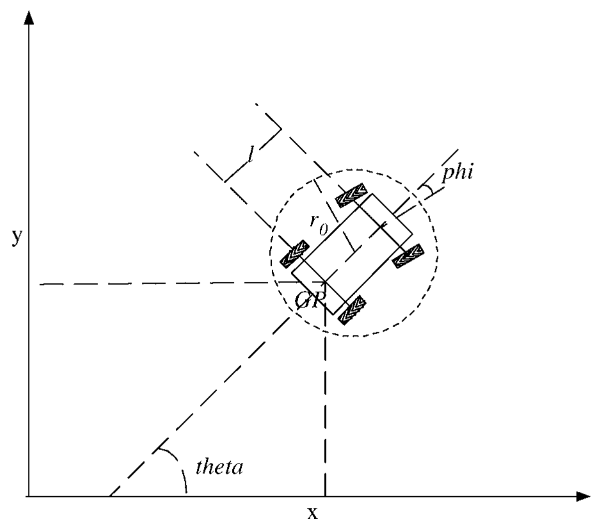

Fig. 2 A car-like vehicle.

(or else by applying by a state transformation), we know that its kinematics model is given by:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} \rho\cos(\theta) & 0 \\ \rho\sin(\theta) & 0 \\ \dfrac{\rho}{l}\tan(\phi) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{5}$$

It is worth noting that the above model is to illustrate the proposed trajectory planning framework. Should the vehicle's kinematic model be different, the proposed idea would still be applicable. It can be shown that the class of polynomial-parameterized trajectories in subsection C are applicable to all vehicles whose kinematic constraints can be transferred into the chain form.[18]

As to interaction with its environment, the vehicle has a physical envelope which is assumed to be a circle centered at $O(t) = (x, y)$ and of radius r_0, and it is equipped with sensors to detect all the obstacles in its sensing range which is assumed to be a circle of radius R_s with $R_s > r_0$.

B. Dynamically Changing Environment

Mathematically, a dynamically changing environment can be described by a sequence of piecewise-constant environments, that is, there exists a sampling period $T_s > 0$ such that, within each time interval $t \in [t_0 + kT_s,\ t_0 + (k+1)T_s)$ where $k = 0, 1, 2, \ldots, \frac{T}{T_s}$ and T/T_s is an integer, velocities of obstacles detected are constants. In this paper, superscript ^{k} and subscript $_k$ both are the index for the kth sampling period, for instance, $t_k = t_0 + kT_s$. The proposed mathematical abstraction of a changing environment consists of the following:

- Vehicle: In addition to its physical model (to be stated shortly), it has a limited sensing range (denoted by a circle of radius R_s) and two-dimensional operational velocity $v(t) \triangleq [\ \dot{x},\ \dot{y}\ ]^{\mathrm{T}}$. Physical envelope of the vehicle is assumed to be a circle centered at $O(t) = (x, y)$ and of radius r_0.
- Obstacles: For some nonnegative integer N_o (which may be unknown), the ith obstacle (for any $i \in \{1, \ldots, N_o\}$) has a circular envelope centered at $O_i(t) = (x_i, y_i)$ and of radius r_i, and it moves according to its instantaneous velocity $v_i(t) \triangleq [v_{i,x}, v_{i,y}]$.
- Detection: Motion of obstacles are not modelled, known apriori or predictable. Nonetheless, as soon as an obstacle emerges within the sensing range of the vehicle, the corresponding center location $O_i(t) = (x_i, y_i)$, obstacle radius r_i and motion velocity $v_i(t)$ are all detected. At time t, the number of obstacles detected is denoted by $n_o(t)$ (with $n_o(t) \le N_o$).

- Piecewise-constant evolution: there exists a sampling period T_s such that, within the time interval $t \in [t_0 + kT_s,\ t_0 + (k+1)T_s)$, $v_i(t) = v_i^k$ are constant (for those $n_o(t)$ obstacles seen by the vehicle). Accordingly, trajectory replanning needs to be done continually to take into account new sensing information, for which analytical solution is preferred for real-time implementation. In the latter parts, the subscript or superscript k of all variables mean they are in the k'th sampling period.

Graphically, operation of the vehicle can be represented by Fig. 3, and the above abstraction accounts for most of the practical issues. In addition, sampling period T_s could be allowed to vary over time (except that notations would become more involved).

C. Piecewise Polynomial Parameterization of Trajectories

As described in subsection B, the environment is modeled as piecewise-constant evolution. Therefore, the trajectory needs to be evolved piecewise. To meet the requirement of piecewise evolution, in the kth sampling period, the corresponding boundary conditions of trajectories must be satisfied, which are given as

$$\boldsymbol{q}(t_k) = \boldsymbol{q}_k = [x_k, y_k, \theta_k, \phi_k], \quad \text{and} \quad \boldsymbol{q}(t_f) = \boldsymbol{q}_f = [x_f, y_f, \theta_f, \phi_f] \tag{6}$$

where $\boldsymbol{q}_k$ is the configuration variables at the beginning of the kth ($k > 0$) sampling period, determined from the trajectory planned in the previous period, or is the initial condition $\boldsymbol{q}_0 = (x_0, y_0, \theta_0, \phi_0)$.

According to the approximation theorem, trajectories satisfying boundary conditions (6) can be described by the polynomial shown as follows:

$$y = [a_0^k\ a_1^k\ \cdots\ a_p^k][1\ x\ \cdots\ x^p]^{\mathrm{T}} \tag{7}$$

where $p\ (\geq 5)$ is the order of the polynomial. It is clear that the case of $p = 5$ generates a unique trajectory, which is fixed and gives no freedom to avoid obstacles. $p > 5$ yields a family of trajectories, thus collision-free paths can be chosen from them. Based on kinematic Eq. (5), assuming the total steering time T is given, the general class of trajectories under boundary conditions (6) and the corresponding control inputs are obtained according to the following lemma.

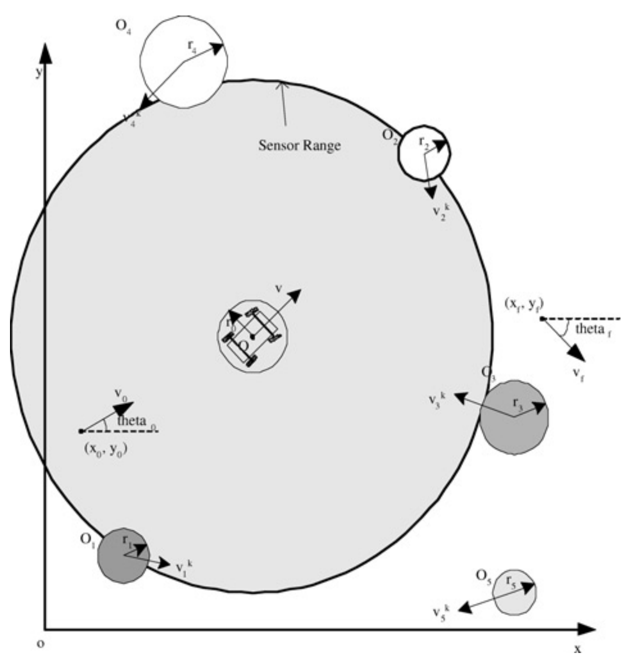

Fig. 3 Operation of a vehicle with limited sensing range.

Lemma 1. *Consider a mobile vehicle given by (5). Under the boundary conditions (6), for $t \in (t_0 + kT_s, t_0 + (k+1)T_s]$, the family of the mobile vehicle's trajectories can be parameterized as follows*[16]*:*

$$y = X[a_0^k, a_1^k, a_2^k, a_3^k, a_4^k, a_5^k]^{\mathrm{T}} + a_6^k x^6 \tag{8}$$

and is achievable under the following steering controls:

$$u_1^k = \frac{w_1^k}{\rho \cos(\theta_k)} \tag{9}$$

and

$$u_2^k = -\frac{3\sin(\theta_k)}{l\cos^2(\theta_k)} \sin^2(\phi_k) w_1^k + l\cos^3(\theta_k)\cos^2(\phi_k) w_2^k \tag{10}$$

where a_6^k is a free design parameter and

$$\begin{aligned}
X &= [1\ x\ x^2\ x^3\ x^4\ x^5] \\
[a_0^k, a_1^k, a_2^k, a_3^k, a_4^k, a_5^k]^{\mathrm{T}} &= (B^k)^{-1}(Y^k - A^k a_6^k) \\
A^k &= [(x^k)^6\ \ 6(x^k)^5\ \ 30(x^k)^4\ \ (x_f)^6\ \ 6(x_f)^5\ \ 30(x_f)^4]^{\mathrm{T}}
\end{aligned}$$

$$Y^k = \begin{bmatrix} y^k \\ \tan(\theta^k) \\ \dfrac{\tan(\phi^k)}{l\cos^3(\theta^k)} \\ y_f \\ \tan(\theta_f) \\ \dfrac{\tan(\phi_f)}{l\cos^3(\theta_f)} \end{bmatrix} \quad B^k = \begin{bmatrix} X|_{x=x^k} \\ \dfrac{\partial X}{\partial x}|_{x=x^k} \\ \dfrac{\partial^2 X}{\partial x^2}|_{x=x^k} \\ X|_{x=x_f} \\ \dfrac{\partial X}{\partial x}|_{x=x_f} \\ \dfrac{\partial^2 X}{\partial x^2}|_{x=x_f} \end{bmatrix}$$

$$\begin{aligned}
w_1^k &= \frac{x_f - x_k}{t_f - t_k}, \\
w_2^k &= 6[a_3^k + 4a_4^k x_1^k + 10a_5^k (x_1^k)^2 + 20a_6^k (x_1^k)^3] w_1 \\
&\quad + 24[a_4^k + 5a_5^k x_1^k + 15a_6^k (x_1^k)^2](t - t_0 - kT_s) w_1^2 \\
&\quad + 60(a_5^k + 6a_6^k x_1^k)(t - t_0 - kT_s)^2 w_1^3 \\
&\quad + 120a_6^k (t - t_0 - kT_s)^3 w_1^4.
\end{aligned}$$

Proof. Recalling Eq. (5), it is straightforward to obtain that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \tan\theta, \quad \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{\tan(\phi)}{l\cos^3(\theta)} \tag{11}$$

Taking advantage of this characteristic, we can satisfy both vehicle's kinematic model (5) and boundary condition (6) by parameterized polynomials of sufficient order. Actually, to ensure there exist a solution, at least a fifth order polynomial should be applied. Choosing a six order polynomial of y in terms of x, shown as Eq. (8), it is easy to obtain all the results according to Eq. (11). ■

Since $p = 5$ is the simplest case to achieve a family of trajectories, we discuss it to present the framework of avoiding obstacles. Thus, only one adjustable parameter a_6^k is kept and used to generate trajectories. Any collision avoidance criterion for obstacles can be mapped into the domain of a_6^k, therefore feasible trajectories are obtained.

In each sampling(evolution) period, the piecewise sixth order polynomial obtained from lemma 1 may not represent arbitrary trajectories well. Also, since $w_1 = (x_f - x_k)/(t_f - t_k)$, the obtained trajectory can only move in one direction along the x axis, which means the obtained trajectory is a single-valued function. However, given waypoints, each pair of neighbor points (including original boundary points and waypoints) and their associated steering angles are treated as new boundary conditions, and Lemma 1 is reapplied to generate sub-trajectories. The obtained series of sixth order polynomials (like interpolation) are still applicable to describe any trajectories. These waypoints are given by the higher level planner according to requirements. For instance, based on static maps, typical searching approaches such as A^* could be used to find these waypoints by using some prior knowledge.

D. Optimal Performance Index for Trajectory Planning

To obtain an optimal solution from the family of trajectories, an applicable performance index (3), such as the length of path, should be assigned. The method to obtain the shortest trajectory is to compute the minimum value of the line integral with respect to arc length. The solution can only be obtained numerically. However, for applications in dynamical environments, the optimal trajectory should be obtained in real-time. Thus, an analytical solution is preferred. In what follows, we present a performance index in the form of L_2 norm such that an optimal trajectory, analogous to the shortest path, will be achieved. Moreover, the optimal solution is analytical. Intuitively, without taking into account kinematic constraints, it is well known that the shortest trajectory from the starting point to the end point is the beeline between them, called “initial straight line”. Hence, if the feasible trajectory for the mobile vehicle, where kinematic constraints are considered, can be made to stay as close as possible to this “initial straight line” in real-time, then it’s reasonable to say that the trajectory is shortened effectively. To this end, let the cost function be formulated according to the integral of the square of the difference between the vertical coordinate of the parameterized trajectories and that of the “initial straight line”. It is graphically illustrated in the Fig. 4. To match the piecewise-constant evolution in the kth sampling period, the cost function is rewritten as

$$J_k = \int_{x_k}^{x_f} \left[y - \frac{y_f - y_0}{x_f - x_0}(x - x_0) - y_0 \right]^2 \mathrm{d}x \tag{12}$$

Thus, for this paper, the optimal problem is to choose feasible trajectories (8) which minimizes (12).

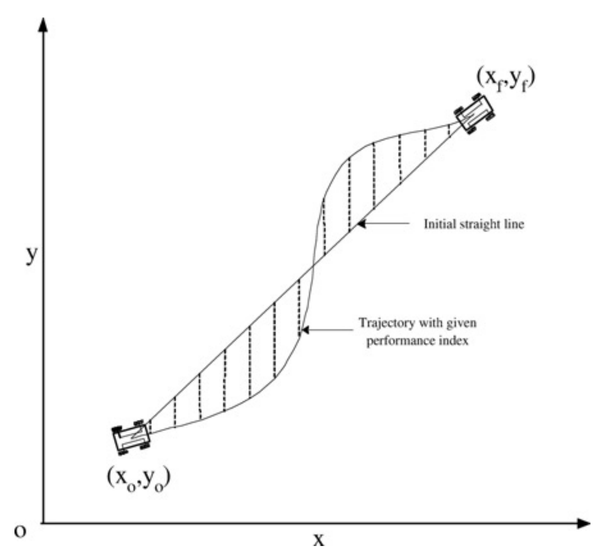

Fig. 4 The performance index and the initial straight line.

E. Collision Avoidance Criteria for Different Type of Obstacles

Generally, two kinds of obstacles exist in the environment, that is, "hard" obstacles and "soft" obstacles. Their definitions are shown as follows:

- ***Hard obstacle***: The obstacle should be avoided by the vehicle and the avoidance is mandatory, such as human beings, other vehicles and equipment in the environment.
- ***Soft obstacle***: The obstacle is preferred to be avoided but can be run over/through by the vehicle, such as unpaved or gravel areas, potholes, and "splash"and "spray"on the wet road generated from other vehicle's wheels.

To be free of collision, the trajectory will be of the distance at least $(r_0 + r_i)$ from the ith obstacle. Using the relative velocity, the collision avoidance criterion should only be considered for $x_i' \in [\underline{x}_i', \ \overline{x}_i']$, with $\underline{x}_i' = x_i^k - r_i - r_0$ and $\overline{x}_i' = x_i^k + r_i + r_0$. Assume the velocity of the obstacle is constant in each sampling period, the criterion for "hard" obstacle is

$$(y - y_i^k - v_{i,y}^k \tau)^2 + (x - x_i^k - v_{i,x}^k \tau)^2 \geq (r_i + r_0)^2 \tag{13}$$

where $\tau = t - t_k$ for $t \in [t_k, t_f]$.

Substituting (8) into (13) leads to

$$\min_{t \in [\underline{t}_i^*, \overline{t}_i^*]} G_i(t, \tau, a_6^k) \geq 0 \tag{14}$$

where

$$\underline{t}_i^* = t_0 + kT_s + \frac{x_i^k - x^k - r_i - r_0}{V_x^k - v_{i,x}^k}$$
$$\overline{t}_i^* = t_0 + kT_s + \frac{x_i^k - x^k + r_i + r_0}{V_x^k - v_{i,x}^k}$$
$$\tau = t - t_0 - kT_s$$
$$G_i(t, \tau, a_3^k) = g_2(x(t), k)(a_3^k)^2 + g_{1,i}(x(t), k, \tau)a_3^k + g_{0,i}(x(t), k, \tau)$$

$[\underline{t}_i^*, \overline{t}_i^*]$ is the time interval (if exists) during which a collision may happen, and

$$g_2(x(t), k) = \left[(x(t))^6 - X(B^k)^{-1}A^k\right]^2$$
$$g_{1,i}(x(t), k, \tau) = 2\left[(x(t))^6 - X(B^k)^{-1}A^k\right] \cdot \left[X(B^k)^{-1}Y^k - y_i^k - v_{i,y}^k \tau\right]$$
$$g_{0,i}(x(t), k, \tau) = \left[X(B^k)^{-1}Y^k - y_i^k - v_{i,y}^k \tau\right]^2 + (x(t) - x_i^k - v_{i,x}^k \tau)^2 - (r_i + r_0)^2$$

Clearly, Analyzing the characteristic of G_i, we have

$$\Omega_{O,i} \triangleq \left\{ a_6^k : \min_{t \in [\underline{t}_i^*, \overline{t}_i^*]} G_i(t, \tau, a_6^k) \geq 0 \right\} = (-\infty, \underline{a}_{6,O,i}^k] \cup [\overline{a}_{6,O,i}^k, +\infty) \tag{15}$$

where

$$\underline{a}_{6,O,i}^k = \max_{x \in [x_0, x_f]} \frac{-g_{1,i} - \sqrt{(g_{1,i})^2 - 4g_2 g_{0,i,j}^l}}{2g_2}$$
$$\overline{a}_{6,O,i}^k = \min_{x \in [x_0, x_f]} \frac{-g_{1,i} + \sqrt{(g_{1,i})^2 - 4g_2 g_{0,i,j}^l}}{2g_2} \tag{16}$$

See[16] for details regarding the case when $g_2 = 0$. Solution (15) is the collision-free interval of a_6^k for the ith obstacle in the environment. Since the "hard" obstacle must be avoided, the criterion for avoiding "hard" obstacles will be exactly the same as the collision-free criterion (14). Assume there are n_H “hard"obstacles in the sensor range. For each “hard” obstacle, in terms of adjustable parameter a_6^k, the collision-free interval is

$$\Omega_{H,l} \triangleq \Omega_{O,l}, \quad l = 1, 2, \ldots, n_H \tag{17}$$

Thus, the collision-free interval for all “hard” obstacles is

$$\Omega_H \triangleq \bigcap_{l=1}^{n_H} \Omega_{H,l} \tag{18}$$

Choosing a_6^k from Ω_H and applying Lemma 1, for “hard” obstacles, we obtain the family of collision-free trajectories, the baseline for avoiding obstacles.

Since “hard” obstacles must be avoided, we assume the optimal performance index for avoiding all “hard” obstacles is J_H^*, and treat it as a baseline. The “soft” obstacles could be taken into accounted by relaxing the performance of this baseline, that is

$$J_S^* \leq \gamma J_H^*$$

where γ is an adjustable relaxation parameter, reflecting the degree of desirability to avoid “soft” obstacles.

In the next section, by using the property of piecewise and parameterized trajectories, the collision avoidance criterion for obstacles is mapped into collision-free intervals of the adjustable parameter, a_6^k, which gives the ability to find the optimal and real-time trajectory.

III. Optimal Solution

As described in Section II, for representing the family of trajectories, the sixth order polynomial is the simplest case, thus it is employed to present the framework of trajectory planning. In this section, collision-free trajectories are obtained and expressed by one adjustable parameter, a_6^k, mapped from collision avoidance criterion. By combining with the performance index (12), an analytical and optimal feasible trajectory will be obtained.

A. Optimal Feasible Trajectory for “Hard” Obstacles

For “hard” obstacles, the analytical and optimal solution with respect to performance index (12) is stated in the following theorem

Theorem 1. *Consider the mobile vehicle given by (5) moving in an environment with dynamically moving “hard” obstacles, and the family of trajectories are parameterized by (8). Let*

$$\begin{aligned} a_6^{k*} = & \frac{13\left(\partial^2 y/\partial x^2\big|_{x=x^k} + \partial^2 y/\partial x^2\big|_{x=x_f}\right)}{12(x^k - x_f)^4} + \frac{117\left(\partial y/\partial x\big|_{x=x_f} - \partial y/\partial x\big|_{x=x^k}\right)}{10(x^k - x_f)^5} \\ & + \frac{429[(y_f - y_0)/(x_f - x_0)(x^k - x_f) - (y^k - y_f)]}{10(x^k - x_f)^6} \end{aligned} \tag{19}$$

Then, under the boundary conditions (6), for “hard” obstacles, the projection of a_6^{k*} on the set Ω_H generates an optimal collision-free trajectory while minimizing performance function (12). The definition of the projection is

$$P_{\Omega_H}^{a_6^{k*}} \triangleq \{a_6^{k**} \in \Omega_H : ||a_6^{k**} - a_6^{k*}|| \leq ||a_6^k - a_6^{k*}||, \forall a_6^k \in \Omega_H\} \tag{20}$$

Proof. It follows from (12) and (8) that

$$\begin{aligned} J_k(a_6^k) &= \int_{x^k}^{x_f} \left[X(B^k)^{-1}(Y^k - A^k a_6^k) + a_6^k x^6 - (y_f - y_0)(x - x_0)/(x_f - x_0) - y_0\right]^2 \mathrm{d}x \\ &= V_x^k \int_{t_0+kT_s}^{t_f} \left[(h_1)^2(a_6^k)^2 + 2h_1 h_2 a_6^k + (h_2)^2\right] \mathrm{d}t \\ &= f_1(a_6^k)^2 + f_2 a_6^k + f_3 \end{aligned} \tag{21}$$

where

$$\begin{aligned} x(t) &= x^k + V_x^k(t - t_k) \\ h_1(t) &= (x(t))^6 - X(B^k)^{-1}A^k \\ h_2(t) &= X(B^k)^{-1}Y - \frac{y_f - y_0}{x_f - x_0}(x(t) - x_0) - y_0 \end{aligned} \tag{22}$$

and

$$\begin{aligned} f_1 &= V_x^k \int_{t_0+kT_s}^{t_f} (h_1)^2 \,\mathrm{d}t, \quad f_2 = 2V_x^k \int_{t_0+kT_s}^{t_f} h_1 h_2 \,\mathrm{d}t, \\ f_3 &= V_x^k \int_{t_0+kT_s}^{t_f} (h_2)^2 \,\mathrm{d}t \end{aligned} \tag{23}$$

Obviously, $f_1 > 0$ since $x^k \neq x_f$. Thus, $J_k(a_6^k)$ is a second order polynomial in terms of a_6^k, and its minimal value is achieved with the choice of

$$a_6^{k*} = -\frac{f_2}{2f_1} \tag{24}$$

It follows from (23) that

$$\begin{aligned} f_1 &= -\frac{(x^k - x_f)^{12}(t_f - t_0)}{12012}, \\ f_2 &= \frac{(x^k - x_f)^6(t_f - t_0)}{27720}\left[5\left(\left.\frac{\partial^2 y}{\partial x^2}\right|_{x=x^k} + \left.\frac{\partial^2 y}{\partial x^2}\right|_{x=x_f}\right)\right. \\ &\quad \times (x^k - x_f)^2 + 54\left(\left.\frac{\partial y}{\partial x}\right|_{x=x_f} - \left.\frac{\partial y}{\partial x}\right|_{x=x^k}\right)(x^k - x_f) \\ &\quad \left. + 198\frac{(y_f - y_0)(x^k - x_f) - (y^k - y_f)(x_f - x_0)}{x_f - x_0}\right] \end{aligned}$$

Then, it is routine to obtain the analytical expression a_6^{k*} in (19).

In the presence of "hard" obstacles, the objective is to find a_6^k that minimizes cost J_k subject to inequality constraint (14). $G_i(t, \tau, a_6^k)$ and $J_k(a_6^k)$ all are second order polynomials of a_6^k. This means $J_k(a_6^k)$ is the homeomorphism of a_6^k when $a_6^k \in (-\infty, a_6^{k*}]$ or $a_6^k \in [a_6^{k*}, +\infty)$. Analyzing this property, we know the projection of a_6^{k*} on the set Ω_H will generate the minimal J_k. This result means $P_{\Omega_H}^{a_6^{k*}}$ yields the optimal solution. Obviously, the optimal solution a_6^{k**} is an analytical solution. ∎

Remark 1. *The "initial straight line" can be updated dynamically. In each sampling period, the beeline between point* (x_k, y_k) *and point* (x_f, y_f) *is set to be "the initial straight line" for this period, that is, the performance index* *in terms of L_2 norm becomes*

$$J_k(a_6^k) = \int_{x_k}^{x_f} \left[y - \frac{y_f - y_k}{x_f - x_k}(x(t) - x_k) - y_k \right]^2 \mathrm{d}x$$

Similarly, we can obtain an optimal trajectory by minimizing this performance index. Compared to the one generated by a_6^{k**}, it may be smoother and its length sometime turns shorter, but a general conclusion about which one is better cannot be drawn.

Remark 2. *Instead of the "initial straight line", if the trajectory (x^*, y^*), generated by $a_6^k = a_6^{k*}$, is used as the reference trajectory, given the performance in the form of L_2 norm, that is,*

$$J_k^*(a_6^k) = \int_{t_k}^{t_f} \left[(x - x^*)^2 + (y - y^*)^2\right] \mathrm{d}t \tag{25}$$

where

$$\begin{cases} x^*(t) & = (F_k)^{-1} Y_k [1 \ \ t \ \ t^2 \ \ t^3], \\ y^*(x(t)) & = (F_k)^{-1}(Y_k - E_k a_6^{k*})[1 \ \ x \ \ x^2 \ \ x^3 \ \ x^4 \ \ x^5] + a_6^{k*} x^6, \end{cases}$$

the same solution a_6^{k**} as shown in Theorem 1 is achieved.

B. Optimal Feasible Trajectory for All Obstacles

In the environment, "soft" obstacles are expected to be avoided, nevertheless, if the cost for avoiding them, such as trajectory length and energy, is high, the vehicle may travel over them. Thus, the impact of avoiding "soft" obstacles should be taken into account. Such impact effect could be measured by relaxing the optimal performance index for avoiding "hard"obstacles with a given tolerance level. The trajectory obtained in the previous part minimized the performance function while avoiding all "hard" obstacles, and it may run over "soft" obstacles. The impact of avoiding "soft" obstacles should be taken into account. Such impact effect could be measured by the corresponding degradation of performance index. Treated as a reference value the optimal performance index for avoiding "hard" obstacles, with a given tolerance level, the associated relaxation of performance index value is known, thus the "soft" obstacles are assigned different privilege to be shunned or run over. Suppose there are n_S "soft" obstacles in the sensor range, the corresponding relaxation inequality with respect to the performance index for each "soft" obstacles is

$$\Omega_{S1,j} \triangleq \left\{a_6^k : J_k(a_6^k) < \gamma_j J_k(a_6^{k**}), \quad j = 1, 2, \ldots, n_S\right\} \tag{26}$$

where $\gamma_j \geq 1$ refers to the relaxation factor for the jth "soft" obstacle, it is chosen according to application requirements. Its value reflects the desired level of trade-off at which the obstacle should be avoided. The bigger the value, the more the desired level, e.g., if $\gamma_j = +\infty$, the "soft" obstacle is actually a "hard" one. The solution of Eq. (26) is

$$\Omega_{S1,j} = (\underline{a}_{6,S1,j}^k, \overline{a}_{6,S1,j}^k) \tag{27}$$

where

$$\underline{a}_{6,S1,j}^k = \frac{-f_2 - \sqrt{(f_2)^2 - 4f_1\left(f_3 - \gamma_j J_k(a_6^{k**})\right)}}{2f_1}$$

$$\overline{a}_{6,S1,j}^k = \frac{-f_2 + \sqrt{(f_2)^2 - 4f_1\left(f_3 - \gamma_j J_k(a_6^{k**})\right)}}{2f_1} \tag{28}$$

This solution is the tolerance range of a_6^k for the vehicle to avoid the jth "soft" obstacle. Since the tolerance range is known, the criterion for avoiding "soft" obstacle is obtained by determining if "soft" obstacles are in the tolerance range. Therefore, collision-free criterion (13) is reapplied on "soft" obstacles, the collision-free interval for the jth "soft" obstacle is

$$\Omega_{S2,j} \triangleq \left\{ a_6^k : \min_{t \in [\underline{t}_j^*, \overline{t}_j^*]} G_j(t, \tau, a_6^k) \geq 0 \right\} \tag{29}$$

Combing the tolerance range (27) with the collision-free interval (29), the interval of a_6^k for avoiding the jth "soft" obstacle is

$$\Omega_{S,j} = \overline{\Omega}_{S1,j} \bigcap \Omega_{S2,j} \tag{30}$$

where $\overline{\Omega}_{S1,j}$ is the complement set of $\Omega_{S1,j}$.

The collision avoidance interval for all "soft" obstacles is

$$\Omega_S \triangleq \bigcap_{j=1}^{n_S} \Omega_{S,j} \tag{31}$$

Based on the results obtained from the previous parts, considering both kinds of obstacles, we obtain the optimal feasible trajectory in the following theorem.

Theorem 2. *Let the mobile vehicle given by (5) operate in an environment with dynamically moving/static "hard" and "soft" obstacles, and the family of trajectories be parameterized by (8). Given a_6^{k*} defined in (19), under the boundary conditions (6), the projection of a_6^{k*} on the set $\Omega \triangleq \Omega_H \bigcap \Omega_S$ generates an optimal feasible trajectory for both kinds of obstacles. The definition of the projection is:*

$$P_{\Omega}^{a_6^{k*}} \triangleq \{a_6^{k***} \in \Omega : ||a_6^{k***} - a_6^{k*}|| \leq ||a_6^k - a_6^{k*}||, \forall a_6^k \in \Omega\} \tag{32}$$

Proof. The steps to prove this theorem are the same as the ones in Theorem 1. ∎

Remark 3. *Theoretically, envelopes of any obstacles are described by a combination of different size circles. Clearly, if we get the collision criteria for the circles forming the envelop of a certain obstacle, the criterion to avoid this obstacle is obtained by combining these criteria together. Thus, without losing generality, Theorems 1 and 2 can be extended to "hard" and "soft" obstacles with any kinds of shapes. The proof is similar to the one in Theorem 1.*

IV. Simulation

In this section, we apply our algorithm to two simple examples and show that the algorithm offers an effective method to generate a feasible trajectory in environments with "soft" and "hard" obstacles. In these simulations, the "star" denotes the start position and "x" marks the end location, and the vehicle's settings are:

- Vehicle parameters: $r_0 = 1$, $l = 0.8$ and $\rho = 0.2$.
- Boundary conditions: $(x_0, y_0, \theta_0, \phi_0) = (0, 0, \pi/4, 0)$ and $(x_f, y_f, \theta_f, \phi_f) = (17, 10, -\pi/4, 0)$, where $t_0 = 0$ and $t_f = 40$ seconds.
- Sensor range $R_s = 8$, implies that the vehicle has a limited sensor range so the vehicle detects the presence of obstacles intermittently; Sampling Period $T_s = 10$.

In the simulation examples, the scales are the same, obstacles are drawn every 5 seconds. And all quantities conform to a given unit system, for instance, meter, meter per second, etc.

Our first example is in the domain of static obstacles. Position of the vehicle is marked by a circle with radius 1 and positions of obstacles are marked by different size circles. In this example, three scenarios are studied and two static obstacles are presented in each of them. Obstacle 1 is located at [2, 3] and of radius 1. Obstacle 2 is presented at position [9, 4] and its radius is 3.

The simulation results are shown in Figs 5(a), 5(b) and 5(c), respectively. In Fig. 5(a), two obstacles are "hard" obstacles, the optimal a_6^k obtained by our approach is applied to generate the trajectory. The length of the trajectory

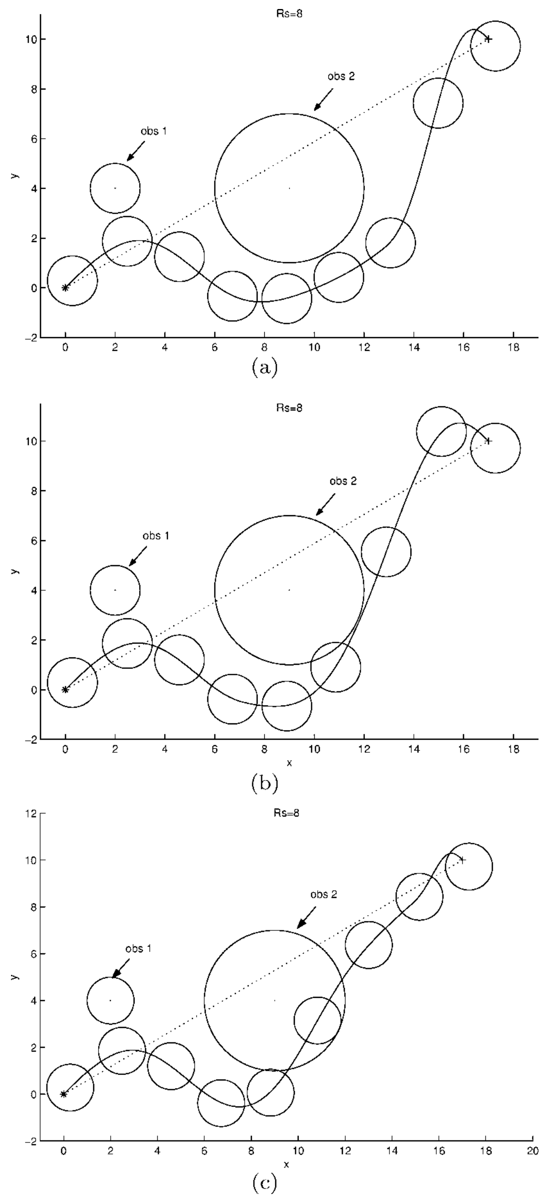

Fig. 5 Comparison of trajectories in a static environment: (a)The optimal trajectory avoiding two "hard" static obstacles; (b)The optimal trajectory avoiding a "hard" obstacle and a "soft" static obstacle with $\gamma = 5$; (c)The optimal trajectory avoiding a "hard" obstacle and a "soft" static obstacle with $\gamma = 3$.

is 24.3. Obstacle 2 is changed to "soft" obstacle in Fig. 5(b), and the relaxation factor $\gamma = 5$. For obstacle 2, the vehicle tries to avoid it at the beginning, however, at last the vehicle chooses to traverse it according to the tolerance range. Thus the distance is shortened to 23.1. In Fig. 5(a), the relaxation factor γ for the "soft" obstacle, the second one, is set to be 3, and the vehicle also runs over it. We find that the trajectory is much closer to the initial straight line, and the trajectory length is reduced to 22.5.

To consider a representative example as a dynamical environment with moving obstacles, we present three cases, in which there exist three moving obstacles. The settings of the vehicle are the same as Example 1. The three obstacles' settings in these two scenarios are shown in Table 1.

Table 1 The settings of three moving obstacles.

	Initial locations of centroid	$t \in [0, 10]$	$t \in [10, 20]$	$t \in [20, 30]$	$t \in [30, 40]$
Obs 1	(5,2)	$v_1^1 = [0, 0.1]$	$v_1^2 = [0.5, 0.2]$	$v_1^3 = [0, -0.2]$	$v_1^4 = [0.1, -0.1]$
Obs 2	(6,4)	$v_2^1 = [-0.5, 0]$	$v_2^2 = [0.2, 0.1]$	$v_2^3 = [0.1, 0.1]$	$v_2^4 = [0.6, 0.1]$
Obs 3	(17,4)	$v_3^1 = [-0.2, -0.1]$	$v_3^2 = [-0.2, 0.1]$	$v_3^3 = [-0.1, 0.1]$	$v_3^4 = [-0.1, 0.1]$

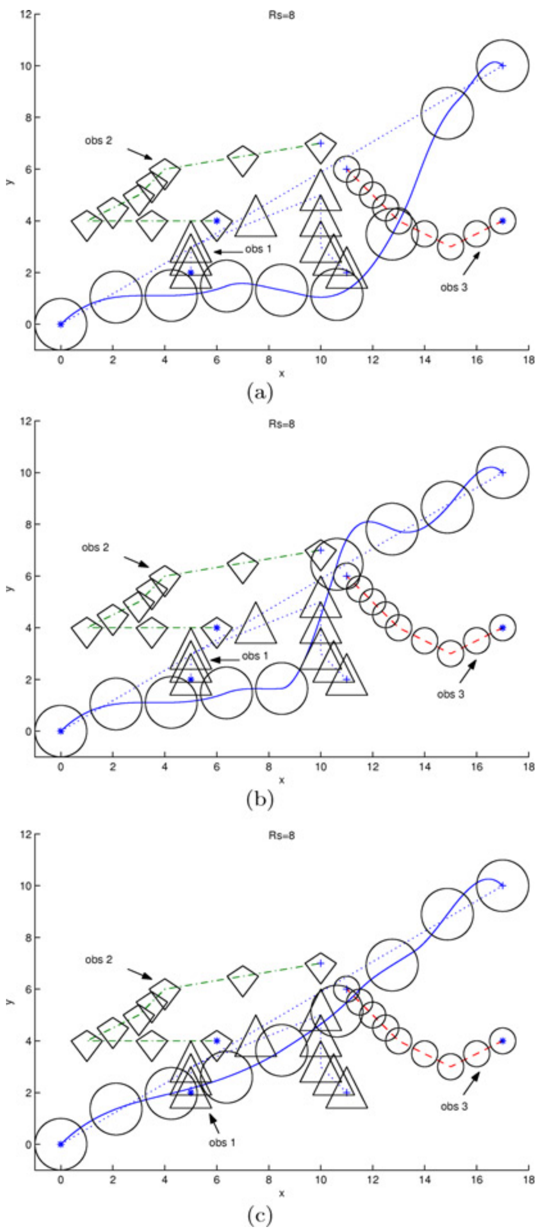

Fig. 6 Comparison of the trajectories in a dynamic environment: (a)The optimal trajectory avoiding three "hard" moving obstacles; (b)The optimal trajectory avoiding two "hard" moving obstacles and one "soft" moving obstacle with relaxation factor $\gamma = 3$; (c)The optimal trajectory avoiding two "hard" moving obstacles and one "soft" moving obstacle with relaxation factor $\gamma = 1$.

The simulation results are shown in Figs 6(a) and 6(b), respectively. In Fig. 6(a), the vehicle uses the optimal trajectory to avoid three "hard" obstacles and the trajectory length is 23.9. For Fig. 6(b), obstacle 1 is changed to be a "soft" one with relaxation factor $\gamma = 3$. In the first two sampling periods, the vehicle also tries to run away from all obstacles, so it follows almost the same trajectory as shown in Fig. 6(a), however, at the beginning of the third sampling period, it finds that the tolerance range will be violated if it shuns all of them, so it traverses the "soft" obstacle 1. Compared to Fig. 6(a), we find that the length of the trajectory in Fig. 6(b) shrinks to 21.7. In Fig. 6(c), obstacle 1 is designed to be a complete "soft" one with relaxation factor $\gamma = 1$, thus, the vehicle will totally ignore obstacle 1 and the length of the path is 20.3.

V. Conclusion

A method for planning the trajectory of a vehicle in a dynamic environment with "hard" and "soft" obstacles has been developed. It is a parameterization approach, since the real-time and feasible trajectory is expressed by a fourth-order parameterized polynomial. The only requirement to obtain the solution of an adjustable parameter makes it possible to generate the realtime trajectory. Key features of this approach are: (1)Parameterized trajectories are employed to satisfy the boundary condition and kinematic equation; (2) Collision-free trajectories are mapped into the associated intervals of the adjustable parameter; (3) An optimal trajectory, with performance index in the form of L_2 norm, for avoiding both kinds of obstacles is analytically obtained.

This approach was illustrated in two examples showing the applicability of the proposed approach to the avoidance of "hard"and "soft" static/moving obstacles. Also demonstrated was the effectiveness of the optimal solution.

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